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ON COMPLETE SEMICARDINAL QUADRATURE FORMULAE

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Abstract—We consider so-called complete semicardinal quadrature formulae (q.f.), as opposed to Euler–Maclaurin or natural semicardinal q.f. By requiring our q.f. to be exact for a particular sequence of B-splines, we are led by a generating function approach to a characterization of these q.f. and are able to determine their coefficients very accurately.

INTRODUCTION

Let m be a natural number and $S_{2m-1}(\boldsymbol{R}^+)$ denote the class of functions $S(x)$ satisfying the two conditions:

$$S(x) \in C^{2m-2}(\boldsymbol{R}^+) \tag{1}$$

$$S(x) \text{ is a polynomial of degree at most } 2m-1 \text{ in each of the intervals } [0,1), (1,2), (2,3), \ldots. \tag{2}$$

The elements of $S_{2m-1}(\boldsymbol{R}^+)$ are the restrictions to $\boldsymbol{R}^+ = [0, \infty)$ of cardinal spline functions of degree $2m-1$.

We can now state

THEOREM 1. *Among all quadrature formulae of the form*

$$\int_0^\infty f(x)\,dx = \sum_{\nu=0}^{\infty} H_\nu^{(m)} f(\nu) + \sum_{j=1}^{m-1} A_j^{(m)} f^{(j)}(0) + Rf \tag{3}$$

where the numerical coefficients $H_\nu^{(m)}$ satisfy the condition

$$H_\nu^{(m)} = 0(1) \text{ as } \nu \to \infty, \tag{4}$$

there is exactly one with the property that

$$Rf = 0 \text{ whenever } f(x) \in S_{2m-1}(\boldsymbol{R}^+) \cap L_1(\boldsymbol{R}^+). \tag{5}$$

This unique q.f. is called *the complete semicardinal quadrature formula of order m*.

From the implication (a proof is similar to that of [1, Lemma 5, Section 9])

$$f(x) \in S_{2m-1}(\boldsymbol{R}^+) \cap L_1(\boldsymbol{R}^+) \text{ implies that } \sum_{\nu=0}^{\infty} |f(\nu)| < \infty, \tag{6}$$

and the condition (4), it follows that the functional Rf is well-defined for all such splines f.

We also mention two other semicardinal formulae: The first is the *Euler–Maclaurin formula*

$$\int_0^\infty f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + \sum_{r=1}^{m-1} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(0) + Rf. \tag{7}$$

The second is what could be called the *natural* semicardinal formula

$$\int_0^\infty f(x)\,dx = \sum_0^\infty \hat{H}_\nu^{(m)} f(\nu) + Rf. \tag{8}$$

Both formulae are uniquely defined among q.f. of their type (i.e., when all their terms are provided with arbitrary coefficients subject only to the condition that the coefficients of $f(\nu)$ should form a bounded sequence) by the condition of being exact; hence $Rf = 0$, whenever $f(x)$ is *any* spline of degree $2m-1$ in the interval $[0,\infty)$, with knots at $1, 2, \ldots$, such that $f(x) \in L_1(\mathbf{R}^+)$. The names of these q.f. come from the names of corresponding interpolation formulae. For a derivation of these q.f. by integrating an appropriate interpolation formula, see [2, Theorem 5, Lecture 8]. For information on the actual forms of Rf in (3), (7), and (8) also see [2, Lecture 8].

Among the q.f. (3), (7), (8), the formula (3) is, as a rule, the most accurate (after an appropriate change of step), while (8) is the least accurate. The computation of the coefficients of the natural formula (8) is the subject of [3]. There, B-splines are used to transform the problem into one that is readily solved by the use of generating functions. Here, we do precisely the same kind of thing, and the entire development is very similar to that of [3].

Our purpose here is to determine the values of the coefficients $H_\nu^{(m)}$, $A_j^{(m)}$ in (3), for $m = 3, 4, 5$ (the cases $m = 6, 7$ are also available). If $m = 1$, the q.f. (3), as well as the q.f. (7), reduces to the natural q.f. (8), and the corresponding q.f. is evidently

$$\int_0^\infty f(x)\,dx = \tfrac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu) + Rf.$$

If $m = 2$, the q.f. (3), as we see at the end of Section 3, becomes

$$\int_0^\infty f(x)\,dx = \tfrac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu) + \tfrac{1}{12}f'(0) + Rf, \tag{9}$$

which is the same as the Euler–Maclaurin q.f. (7) for $m = 2$.

We can also express the q.f. (3) in another way that more explicitly relates this q.f. to the Euler–Maclaurin formula (7).

THEOREM 2. *Let $S(x) \in L_1(\mathbf{R}^+)$ be the unique spline of degree $2m-1$ for $x \geq 0$ with knots $1, 2, \ldots$ satisfying the conditions*

$$S(\nu) = f(\nu) \qquad (\nu = 0, 1, 2, \ldots) \tag{10}$$

$$S^{(i)}(0) = f^{(i)}(0) \qquad (i = 1, 2, \ldots, m-1) \tag{11}$$

Then the q.f. (3) *may be written as*

$$\int_0^\infty f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + \sum_{2i-1 \leq m-1} \frac{B_{2i}}{(2i)!} f^{(2i-1)}(0) + \sum_{2i-1 \geq m} \frac{B_{2i}}{(2i)!} S^{(2i-1)}(0) + Rf$$

A proof follows from observing that the q.f. (7) is exact for the $S(x)$ of the hypothesis and from then employing (10) and (11).

1. B-SPLINES AND EULER-FROBENIUS POLYNOMIALS

We begin our attempt at finding the required coefficients $H_\nu = H_\nu^{(m)}$, $A_j = A_j^{(m)}$ of the q.f. (3) by temporarily ignoring (4) and by enforcing (5) for an appropriate sequence of elements of $S_{2m-1}(\mathbf{R}^+)$. The sequence we require is the sequence of forward B-splines of degree $2m-1$

$$\{Q(x-n)\} \qquad (n = -2m+1, -2m+2, \ldots) \tag{12}$$

where

$$Q(x) = Q_{2m}(x) = \frac{1}{(2m-1)!} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (x-i)_+^{2m-1} \tag{13}$$

in which $x_+ = \max(x, 0)$. We note that $Q(x)$ has integer knots and support in the interval $(0, 2m)$ so that

$$Q(x-n) = 0 \text{ outside the interval } (n, n+2m) \quad (n = -2m+1, -2m+2, \ldots) \tag{14}$$

Evidently $Q(x) \in S_{2m-1}(\boldsymbol{R}^+)$, so we also have

$$Q(x-n) \in S_{2m-1}(\boldsymbol{R}^+) \cap L_1(\boldsymbol{R}^+) \quad (n = -2m+1, -2m+2, \ldots) \tag{15}$$

By substituting $f(x) = Q(x-n)$ in (3) and recalling (5) and (15), we obtain the sequence of relations

$$\int_0^{n+2m} Q(x-n)\,\mathrm{d}x = H_0 Q(-n) + H_1 Q(1-n) + \cdots + H_{n+2m-1} Q(2m+1) + \sum_{j=1}^{m-1} A_j Q^{(j)}(-n) \quad (n = -2m+1, -2m+2, \ldots, -1) \tag{16}$$

and

$$\int_n^{n+2m} Q(x-n)\,\mathrm{d}x = H_{n+1} Q(1) + H_{n+2} Q(2) + \cdots + H_{n+2m-1} Q(2m-1) \quad (n = 0, 1, \ldots) \tag{17}$$

The form of the relations (16), (17) suggests the use of generating functions for the determination of the H_ν and A_j. As in [3], we make use of the close relationship between B-splines and the so-called Euler–Frobenius polynomials. One way to obtain these polynomials that we will denote by $\Pi_k(x)$, as well as some of their properties, is the content of the following: (for a proof, see [4, Lemma 7]).

LEMMA 1. (i) $\Pi_k(x)$ *is a reciprocal, monic polynomial of degree* $k-1$ *with integer coefficients satisfying the recurrence relation*

$$\Pi_{k+1}(x) = (1+kx)\Pi_k(x) + x(1-x)\Pi_k'(x) \quad (\Pi_1(x) = 1) \tag{18}$$

(ii) *The following identity holds*:

$$\Pi_k(x)/(1-x)^{k+1} = \sum_{\nu=0}^{\infty} (\nu+1)^k x^\nu \quad (|x| < 1) \tag{19}$$

(iii) *The zeros* λ_ν *of* $\Pi_k(x)$ *are all simple and negative. In particular, we label the zeros of* $\Pi_{2m-1}(x)$ *so that they satisfy the conditions*

$$\lambda_{2m-2} < \cdots < \lambda_m < -1 < \lambda_{m-1} \cdots < \lambda_1 < 0 \tag{20}$$

and

$$\lambda_1 \lambda_{2m-2} = \lambda_2 \lambda_{2m-3} = \cdots = \lambda_{m-1}\lambda_m = 1. \tag{21}$$

B-splines and the Euler–Frobenius polynomials are tied together by the following result that is perhaps of independent interest [5, Theorem 5, p. 22]:

THEOREM 3. *The following identity holds*:

$$\sum_{\nu=0}^{2m-2} Q^{(j)}(\nu+1)x^\nu = (1-x)^j \Pi_{2m-1-j}(x)/(2m-1-j)! \quad (j = 0, 1, \ldots, m-1;\ m = 1, 2, \ldots). \tag{22}$$

The polynomials $\Pi_{2m-1}(x)$ and their zeros for $m = 2$ through $m = 7$ are listed in [3, Section 7].

2. THE SUMMATION OF CERTAIN SERIES

(A) The right side of (16) and (17) is equal to the coefficient of x^{n+2m+1} in

$$\left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\sum_{\nu=0}^{2m-2} Q(2m-1-\nu)x^\nu\right) + \sum_{\nu=1}^{2m-2}\left[\sum_{j=1}^{m-1} A_j Q^{(j)}(2m+1-\nu)\right]x^\nu. \tag{23}$$

In order to simplify the two polynomials in (23), we note that

$$Q^{(j)}(x) = (-1)^j Q^{(j)}(2m-x) \quad (j=0,1,\ldots,m-1) \tag{24}$$

as can be verified from (13). With this substitution and the interchange of the order of summation in the second polynomial, (23) becomes

$$\left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\left(\sum_{\nu=0}^{2m-2} Q(\nu+1)x^\nu\right) + \sum_{j=1}^{m-1}(-1)^j A_j\left(\sum_{\nu=0}^{2m-2} Q^{(j)}(\nu+1)x^\nu\right), \tag{25}$$

and then, by incorporating (22), becomes

$$\left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\frac{\Pi_{2m-1}(x)}{(2m-1)!} + \sum_{j=1}^{m-1}\frac{A_j(x-1)^j\Pi_{2m-1-j}(x)}{(2m-1-j)!} \tag{26}$$

(B) We now turn our attention to the left side of (16) and (17) and define

$$F_{n+2m-1} = \begin{cases} \displaystyle\int_0^{n+2m} Q(x-n)\,dx & n=-2m+1,\ldots,-1 \\ \displaystyle\int_n^{n+2m} Q(x-n)\,dx & n=0,1,2,\ldots \end{cases} \tag{27}$$

and wish to sum the series

$$\sum_{n=-2m+1}^{\infty} F_{n+2m-1}x^{n+2m-1}.$$

Because $Q(x-n)$ is a B-spline of degree $2m-1$, we can employ

$$\int_n^{n+2m} Q(x-n)\,dx = 1$$

to obtain

$$\sum_{n=-2m-1}^{\infty} F_{n+2m-1}x^{n+2m-1} = \sum_{\nu=0}^{\infty} x^\nu - \sum_{n=-2m+1}^{-1}\left[\int_n^0 Q(x-n)\,dx\right]x^{2m-1+n} \tag{28}$$

Because of the symmetry of $Q(x-n)$ and by a change of variable $y=x-2n-2m$, we find that

$$\int_n^0 Q(x-n)\,dx = \int_{2n+2m}^{n+2m} Q(x-n)\,dx = \int_0^{-n} Q(y+n+2m)\,dy \quad (n=-2m+1,\ldots,-1). \tag{29}$$

By a second change of variable $r=2m-1+n$, we obtain

$$\sum_{n=-2m+1}^{-1}\left[\int_n^0 Q(x-n)\,dx\right]x^{2m-1+n} = \sum_{r=0}^{2m-2}\left[\int_0^{2m-1-r} Q(y+r+1)\,dy\right]x^r. \tag{30}$$

Substituting (30) in (28), we have

$$\sum_{\nu=0}^{\infty} F_\nu x^\nu = \frac{1}{1-x}\left\{1-(1-x)\sum_{r=0}^{2m-2}\left[\int_0^{2m-1-r} Q(y+r+1)\,dy\right]x^r\right\} \tag{31}$$

To simplify the polynomial of degree $2m-1$ in the numerator of the right side of (31) we employ the following lemma, whose proof is similar to that of Theorem 3 and so may be omitted.

LEMMA 2. *The following identity holds*:

$$1-(1-x)\sum_{r=0}^{2m-2}\left[\int_0^{2m-1-r} Q(y+r+1)\,\mathrm{d}y\right]x^r = \frac{\Pi_{2m}(x)}{(2m)!} \quad (m=1,2,\ldots)$$

Using Lemma 2, we may therefore write (31) as

$$\sum_{\nu=0}^{\infty} F_\nu x^\nu = \frac{\Pi_{2m}(x)}{(2m)!(1-x)} \tag{32}$$

Equating the relations in (26) and (32) we see that we require

$$\frac{\Pi_{2m}(x)}{(2m)!(1-x)} = \left(\sum_{\nu=0}^{\infty} H_\nu x^\nu\right)\frac{\Pi_{2m-1}(x)}{(2m-1)!} + \sum_{j=1}^{m-1}\frac{A_j(x-1)^j\Pi_{2m-1-j}(x)}{(2m-1-j)!} \tag{33}$$

Solving (33) for $\sum\limits_{\nu=0}^{\infty} H_\nu x^\nu$ gives the final relation

$$\sum_{\nu=0}^{\infty} H_\nu x^\nu = \frac{\Pi_{2m}(x)}{(2m)(1-x)\Pi_{2m-1}(x)} - \sum_{j=1}^{m-1} A_j\frac{(2m-1)!}{(2m-1-j)!}\frac{(x-1)^j\Pi_{2m-1-j}(x)}{\Pi_{2m-1}(x)} \tag{34}$$

Our derivation of (34) establishes the following:

The coefficients H_ν, A_j of the most general functional

$$Rf = \int_0^\infty f(x)\,\mathrm{d}x - \sum_0^\infty H_\nu f(\nu) - \sum_{j=1}^{m-1} A_j f^{(j)}(0) \tag{35}$$

vanishing for the functions

$$Q_{2m}(x-n) \quad (n=-2m+1,-2m+2,\ldots) \tag{36}$$

are the expansion coefficients of the rational function (34) *where the* A_j $(j=1,\ldots,m-1)$ *are chosen arbitrarily.*

3. DETERMINING THE COEFFICIENTS $H_\nu^{(m)}$, $A_j(m)$.

This will be done by requiring the coefficients H_ν of (35) to satisfy (4) or

$$H_\nu = 0(1) \quad \text{as } \nu\to\infty. \tag{37}$$

Let $R_m(x)$ denote the right side of (34). If we substitute the recurrence relation (18) for $k=2m-1$ in (34), we obtain

$$R_m(x) = \frac{1+(2m-1)x}{2m(1-x)} + \frac{x\Pi'_{2m-1}(x)}{2m\Pi_{2m-1}(x)} - \sum_{j=1}^{m-1} A_j\frac{(2m-1)!}{(2m-1-j)!}\frac{(x-1)^j\Pi_{2m-1-j}(x)}{\Pi_{2m-1}(x)} \tag{38}$$

where the A_j's are as yet undetermined. Because of (20), the poles of $R_m(x)$ are simple, and we can readily decompose $R_m(x)$ into partial fractions. Observing that the last two expressions in the right side of (38) are regular at ∞, where they assume the values $1-(1/m)$ and

$$\sum_{j=1}^{m-1} A_j\frac{(2m-1)!}{(2m-1-j)!}$$

respectively, we find that

$$R_m(x) = -\frac{1}{2m} - \sum_{j=1}^{m-1} A_j \frac{(2m-1)!}{(2m-1-j)!} + \frac{1}{1-x} + \frac{1}{2m} \sum_{\nu=1}^{2m-2} \frac{\lambda_\nu}{x-\lambda_\nu}$$
$$- \sum_{\nu=1}^{2m-2} \sum_{j=1}^{m-1} A_j \frac{(2m-1)!}{(2m-1-j)!} \frac{(\lambda_\nu - 1)^j \Pi_{2m-1-j}(\lambda_\nu)}{\Pi'_{2m-1}(\lambda_\nu)(x-\lambda_\nu)} \tag{39}$$

Observing that the poles $\lambda_1, \ldots, \lambda_{m-1}$ are inside the unit circle while $\lambda_m, \ldots, \lambda_{2m-2}$ are outside, in view of (20), we can satisfy (37) if and only if the coefficients A_j can be chosen so that the $m-1$ poles $\lambda_1, \ldots, \lambda_{m-1}$ of $R_m(x)$ have vanishing residues. By (39), this occurs if and only if the A_j satisfy the equations

$$\sum_{j=1}^{m-1} A_j \frac{(\lambda_\nu - 1)^j \Pi_{2m-1-j}(\lambda_\nu)}{(2m-1-j)!} = \frac{\lambda_\nu \Pi'_{2m-1}(\lambda_\nu)}{(2m)!} \quad (\nu = 1, \ldots, m-1) \tag{40}$$

And, in fact, (40) determines the A_j uniquely because

$$\det\left(\frac{(\lambda_\nu - 1)^j \Pi_{2m-1-j}(\lambda_\nu)}{(2m-1-j)!}\right) \neq 0 \quad (\nu = 1, \ldots, m-1;\ j = 1, \ldots, m-1)$$

(For a proof, see [5, p. 36–37] for $I \cup I' = \{1, 2, \ldots, m-1\}$). This establishes

THEOREM 4. *There is a unique q.f.*

$$\int_0^\infty f(x)\,dx = \sum_{\nu=0}^{\infty} H_\nu^{(m)} f(\nu) + \sum_{j=1}^{m-1} A_j^{(m)} f^{(j)}(0) + Rf \tag{41}$$

that vanishes for the sequence of B*-splines* $Q(x-n)$ $(n = -2m+1,\ -2m+2, \ldots)$ *and whose coefficients* $H_\nu^{(m)}$ *are bounded. These coefficients are generated by the expansion*

$$R_m(x) = \sum_{\nu=0}^{\infty} H_\nu^{(m)} x^\nu \quad (|x| < 1) \tag{42}$$

where

$$R_m(x) = -\frac{1}{2m} - \sum_{j=1}^{m-1} A_j \frac{(2m-1)!}{(2m-1-j)!} + \frac{1}{1-x}$$
$$+ \sum_{\nu=m}^{2m-2} \left\{ \frac{\lambda_\nu}{2m} - \sum_{j=1}^{m-1} A_j \frac{(2m-1)!}{(2m-1-j)!} \frac{(\lambda_\nu - 1)^j \Pi_{2m-1-j}(\lambda_\nu)}{\Pi'_{2m-1}(\lambda_\nu)} \right\} \frac{1}{x-\lambda_\nu} \tag{43}$$

and the A_j *are defined by* (40).

To complete a proof of Theorem 1 we must still show that $Rf = 0$ if $f(x) \in S_{2m-1}(R^+) \cap L_1(\mathbf{R}^+)$. This is proved in much the same way as Theorem 1 of [6, Section 5], and so we omit it.

In order to actually compute the coefficients H_ν, it is better to expand the right side of (43) in terms of $\lambda_1, \ldots, \lambda_{m-1}$ since these zeros are all less than one in magnitude. Using (21) and the relations

$$\Pi'_{2m-1}(\lambda_\nu^{-1}) = -\lambda_\nu^{2m+4} \Pi'_{2m-1}(\lambda_\nu) \quad (\nu = 1, \ldots, m-1) \tag{44}$$

and

$$\Pi_{2m-1-j}(\lambda_\nu^{-1}) = \lambda_\nu^{-2m+2+j} \Pi_{2m-1-j}(\lambda_\nu) \quad (\nu = 1, \ldots, m-1), \tag{45}$$

we can write the last term on the right side of (43) as

$$\sum_{\nu=1}^{m-1}\left(-\frac{1}{\lambda_\nu}\right)\frac{(2m-1)!}{\Pi'_{2m-1}(\lambda_\nu)}\left\{\frac{\lambda_\nu\Pi'_{2m-1}(\lambda_\nu)}{(2m)!}-\sum_{j=1}^{m-1}(-1)^{j+1}A_j\frac{(\lambda_\nu-1)^j\Pi_{2m-1-j}(\lambda_\nu)}{(2m-1-j)!}\right\}\frac{1}{1-\lambda_\nu x}. \tag{46}$$

By substituting the relations (40) in (46), we see that the sum in (46) involves A_j with only even subscripts, so that (43) becomes

$$R_m(x)=C+\frac{1}{1-x}+\sum_{\nu=1}^{m-1}C_\nu\frac{1}{1-\lambda_\nu x} \tag{47}$$

where

$$C=-\frac{1}{2m}-\sum_{j=1}^{m-1}A_j\frac{(2m-1)!}{(2m-1-j)!} \tag{48}$$

and

$$C_\nu=-\frac{2(2m-1)!}{\lambda_\nu\Pi'_{2m-1}(\lambda_\nu)}\sum_{2j<m}A_{2j}\frac{(\lambda_\nu-1)^{2j}\Pi_{2m-1-2j}(\lambda_\nu)}{(2m-1-2j)!}\quad(\nu=1,\ldots,m-1), \tag{49}$$

and the A_j are gotten from (40).

Expanding the right side of (47) in powers of x and using (42), we obtain

Corollary 1. *The coefficients of the q.f.* (3) *have the values*

$$H_0^{(m)}=C+1+\sum_{\nu=1}^{m-1}C_\nu, \tag{50}$$

$$H_k^{(m)}=1+\sum_{\nu=1}^{m-1}C_\nu\lambda_\nu^{\,k}\quad(k=1,2,\ldots), \tag{51}$$

where C and C_ν are given by (48) *and* (49).

The case $m=2$. We mention this case separately because the results are explicit. We note that $\Pi_2(x)=x+1$ and

$$\Pi_3(x)=x^2+4x+1 \text{ so that } \lambda_1=-2+\sqrt{3}.$$

From the expressions (40) and (48)–(51), we find that

$$A_1^{(2)}=\tfrac{1}{12},\qquad C=-\tfrac{1}{2},\qquad C_1=0$$

and therefore

$$H_0^{(2)}=\tfrac{1}{2},\quad H_j^{(2)}=1\qquad(j=1,2,\ldots),$$

so that we get the q.f. (9).

4. NUMERICAL VALUES OF THE COEFFICIENTS

Instead of listing the coefficients $H_\nu^{(m)}$ ($\nu=0,1,\ldots$) directly, we find it more convenient to define

$$h_0^{(m)}=H_0^{(m)}-\tfrac{1}{2},\quad h_k^{(m)}=H_k^{(m)}-1\qquad(k=1,2,\ldots)$$

and to list these. The q.f. (3) then becomes

$$\int_0^\infty f(x)\,dx=T+\sum_{\nu=0}^{\infty}h_\nu^{(m)}f(\nu)+\sum_{j=1}^{m-1}A_j^{(m)}f(0)+Rf$$

where T stands for the trapezoidal sum

$$T = \tfrac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu).$$

By using (50) and (51), we obtain

$$h_0^{(m)} = C + \tfrac{1}{2} + \sum_{\nu=1}^{m-1} C_\nu$$

and

$$h_k^{(m)} = \sum_{\nu=1}^{m-1} C_\nu \lambda_\nu^k \qquad (k = 1, 2, \ldots).$$

The λ_ν, the zeros of the Euler–Frobenius polynomials, are listed in [3, Section 7] for each m. All the computations are carried out in double precision and all decimals listed should be correct.

$m = 3$: $A_1 = 0.11029\,24726$
$A_2 = 0.00926\,93793\,17$

k	$10^{10} \cdot h_k^{(3)}$	k	$10^{10} \cdot h_k^{(3)}$	k	$10^{10} \cdot h_k^{(3)}$	k	$10^{10} \cdot h_k^{(3)}$
0	327 414 433	6	3 611 667	12	23 014	18	147
1	−407 624 015	7	−1 555 085	13	−9 909	19	−63
2	112 127 019	8	669 581	14	4 267	20	27
3	−45 547 436	9	−288 305	15	−1 837	21	−12
4	19 493 877	10	124 137	16	791	22	5
5	−8 388 509	11	−53 450	17	−341	23	−2
						24	1

$m = 4$: $A_1 = 0.11283\,27603$
$A_2 = 0.01038\,99221\,7$
$A_3 = 0.00027\,15540\,358$

k	$10^{10} \cdot h_k^{(4)}$	k	$10^{10} \cdot h_k^{(4)}$	k	$10^{10} \cdot h_k^{(4)}$	k	$10^{10} \cdot h_k^{(4)}$
0	354 789 383	9	−1 206 001	18	4 351	27	−16
1	−438 613 389	10	645 545	19	−2 329	28	8
2	119 774 332	11	−345 547	20	1 247	29	−4
3	−54 138 115	12	184 964	21	−667	30	2
4	27 794 314	13	−99 008	22	357	31	−1
5	−14 732 900	14	52 997	23	−191	32	1
6	7 868 484	15	−28 368	24	102		
7	−4 209 670	16	15 185	25	−55		
8	2 253 087	17	−8 128	26	29		

$m = 5$: $A_1 = 0.09161\,93434\,0$
$A_2 = 0.00232\,21928\,67$
$A_3 = -0.00134\,36478\,75$
$A_4 = -0.00014\,24110\,066$

k	$10^{10} \cdot h_k^{(5)}$	k	$10^{10} \cdot h_k^{(5)}$	k	$10^{10} \cdot h_k^{(5)}$	k	$10^{10} \cdot h_k^{(5)}$
0	108 285 589	10	704 835	20	4 865	30	34
1	−144 620 360	11	−428 522	21	−2 958	31	−20
2	53 043 819	12	260 537	22	1 798	32	12
3	−25 870 791	13	−158 405	23	−1 093	33	−8
4	14 534 813	14	96 310	24	665	34	5
5	−8 600 262	15	−58 556	25	−404	35	−3
6	5 181 328	16	35 602	26	246	36	2
7	−3 140 636	17	−21 646	27	−149	37	−1
8	1 907 563	18	13 161	28	91	38	1
9	−1 159 403	19	−8 002	29	−55		

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